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ON THE HOMOLOGY OF CERTAIN SPACES LOOPED BEYOND THEIR CONNECTIVITY

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ABSTRACT

Very little is known about $H_*(\Omega^n X)$ when n is larger than the connectivity of X. In this paper we calculate this when $X = \Omega^\infty S^\infty$ and $n = 1$ or 2, and when $X = \mathrm{JU}(q)$ or $\mathrm{JSO}(3)$ and n is arbitrary. Some information is also given when X is a sphere.

One approach to studying the homotopy theory of a space X is to consider the n-fold loop space on X where n may be much larger than the connectivity of X. Of course, very little is known about the homology of a space which has been looped beyond its connectivity. In this paper we shall study this problem for certain spaces X.

Write $\Omega_0^n X$ for the component of the base-point in $\Omega^n X$. We shall give an explicit calculation of $H_*\Omega_0^n(\Omega_0^\infty S^\infty)$ for $n = 1$ and 2. Write $\mathrm{JU}(q)$ for the fibre of $\psi^q - 1 : \mathrm{BU} \to \mathrm{BU}$ and $\mathrm{JSO}(3)$ for the fibre of $\psi^3 - 1 : \mathrm{BSO} \to \mathrm{BSO}$. We compute the mod-$p$ homology of $\Omega_0^n\mathrm{JU}(q)$ and the mod-2 homology of $\Omega_0^k\mathrm{JSO}(3)$. One consequence is that if $f: X \to \mathrm{JU}(q)$ is a map which is a split epimorphism on the first non-vanishing homotopy group of $\mathrm{JU}(q)$ localized at the "usual" primes (defined after 1.4), then $H_*\Omega_0^n X$ has a primitively generated sub-Hopf algebra which is a polynomial algebra with infinitely many generators for all $n \geqq 2$.

We give some information on $H_*(\Omega_0^{n+k}S^n; F_2)$ for some values of n and k obtained from James' filtration of ΩS^n. All of this is closely related to the

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Whitehead square, $\omega_n = [i_n, i_n]$, $\omega_n : S^{2n-1} \to S^n$, and its generalizations. For example, by adjointing ω_n one gets a map $\bar{\omega}_n : S^{2n-1-q} \to \Omega_0^q S^n$ and one might ask if this map is non-zero in mod-2 homology. This "low-dimensional" information then has global implications for the structure of $H_*(\Omega_0^{n+q}S^n; F_2)$ (at least in the examples where we can compute answers). Thus we give some estimates on q such that $\bar{\omega}_n$ is non-zero on $H_{2n-1-q}(\ ; F_2)$. The situation here is that this last map is non-zero after *relatively* few loops except in the possible Arf invariant cases: $n = 2^k - 1$. We include one specific example here by calculating $H_*(\Omega_0^4 S^3; F_2)$. The mod-2 homology of $\Omega_0^{n+1}S^n$ was computed by Tom Hunter [H].

It seems worthwhile to make the following observations: In the examples where we are able to do explicit calculations, the homology of $\Omega_0^{n+q}\Sigma^n X$ contains a polynomial algebra with infinitely many generators. Furthermore the nilpotent elements have bounded order of nilpotence and this order is a function of q. For example, $H_*(\Omega_0^{n+1}S^n; F_2)$ contains an exterior algebra, but if $x^2 \neq 0$, then $x^t \neq 0$ for all t in our examples.

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We would like to take this opportunity to express our fondness for Alex Zabrodsky both as a mathematician and as a wonderful human being.

1. Statement of results

We restrict attention to the prime 2 for Theorems 1.1 and 1.2.

THEOREM 1.1. *$H^*(\Omega_0^{\infty+1}S^\infty; F_2)$ is isomorphic to a polynomial algebra with primitive generators. Thus $H_*(\Omega_0^{\infty+1}S^\infty; F_2)$ is isomorphic to an exterior algebra as an algebra.*

THEOREM 1.2. *$H^*(\Omega_0^{\infty+2}S^\infty; F_2)$ is isomorphic to a tensor product of polynomial algebras and exterior algebras. All even dimensional generators are polynomial and "most" odd dimensional generators are polynomial.*

We remark that explicit generators in Theorems 1.1 and 1.2 are given in their proofs. The situation is different for $H_*(\Omega_0^{\infty+3}S^\infty; F_2)$.

In the next theorem, p is any prime.

THEOREM 1.3. *There is an isomorphism of algebras*

$$H_*(\Omega_0^{2n}\mathrm{JU}(q); F_p) \to \bigotimes_{p \mid (q^{n+k}-1)} (\Lambda[f_{2k-1}] \otimes F_p[x_{2k}])$$

where f_{2k-1} is the unique primitive in the image of $H_{2k-1}(\Omega_0^{2n+1}\mathrm{BU}; F_p) \to H_{2k-1}(\Omega_0^{2n}\mathrm{JU}(q); F_p)$ and x_{2k} has non-zero image in $H_{2k}(\Omega_0^{2n}\mathrm{BU}; F_p)$.

THEOREM 1.4. *There is an isomorphism*

$$H_*(\Omega_0^{2n-1}\mathrm{JU}(q); F_p) \to \bigotimes_{p \mid (q^{n+k}-1)} (\Lambda[x_{2k+1}] \otimes F_p[x_{2k}])$$

which is one of algebras if $p > 2$ where x_{2k} is in the image of $H_{2k}(\Omega_0^{2n}\mathrm{BU}; F_p) \to H_{2k}(\Omega_0^{2n-1}\mathrm{JU}(q); F_p)$ and x_{2k+1} has non-zero image in $H_{2k+1}(\Omega_0^{2n-1}\mathrm{BU}; F_p)$.

Next fix an odd prime p and choose a prime q such that $q^i - 1 \not\equiv 0(p)$ for $0 < i < p-1$ and $v_p(q^{p-1} - 1) = 1$ (and recall that infinitely many such q exist). Assume that X is an H-space and there is an H-map $f: X \to \mathrm{JU}(q)$ such that f induces a split epimorphism on the p-primary component of $\Pi_{2p-3}\mathrm{JU}(q) \cong Z/p$.

THEOREM 1.5. *Assume that X satisfies the above hypotheses. Then $H_*(\Omega_0^n X; F_p)$ contains a primitively generated Hopf algebra which is polynomial on infinitely many generators if $n \geqq 1$.*

Next, we give the mod-2 (co-)homology of $\Omega_0^k \mathrm{JSO}(3)$.

THEOREM 1.6. *The* mod-2 *(co-)homology of $\Omega_0^k \mathrm{JSO}(3)$ is given as follows:*

(i) *If $k \equiv 1$ mod 8, $H^*\Omega_0^k\mathrm{JSO}(3)$ is isomorphic to $H^*\mathrm{Spin} \otimes H^*\mathrm{SO}/U$ as an algebra.*

(ii) *If $k \equiv 2$ mod 8, $H^*\Omega_0^k\mathrm{JSO}(3)$ is isomorphic to $H^*\mathrm{SO}/U \otimes H^*U/\mathrm{Sp}$ as a vector space.*

(iii) *If $k \equiv 3$ mod 8, $H^*\Omega_0^k\mathrm{JSO}(3)$ is isomorphic to $H^*\mathrm{SU}/\mathrm{Sp} \otimes H^*\mathrm{BSp}$ as an algebra.*

(iv) *If $k \equiv 4$ mod 8, $H^*\Omega_0^k\mathrm{JSO}(3)$ is isomorphic to $H^*\mathrm{BSp} \otimes H^*\mathrm{Sp}$ as a vector space.*

(v) *If $k \equiv 5$ mod 8, $H^*\Omega_0^k\mathrm{JSO}(3)$ is isomorphic to $H^*\mathrm{Sp} \otimes H^*\mathrm{Sp}/U$ as a vector space.*

(vi) *If $k \equiv 6$ mod 8, $H^*\Omega_0^k\mathrm{JSO}(3)$ is isomorphic to $H^*\mathrm{Sp}/U \otimes H^*UO$ as a vector space.*

(vii) *If $k = 8j - 1$, then $H_*\Omega_0^k\mathrm{JSO}(3)$ is isomorphic to*

$$F_2[p_{2n+1}^2 \mid n \equiv 0 \text{ mod } 2^{\nu_j}] \otimes F_2[p_2, p_{2n+1} \mid 2n \not\equiv 0 \text{ mod } 2^{\nu_j}] \otimes F_2[e_n]/e_n^{2^{\nu_j}}$$

as an algebra where the degree of p_i is i, the degree of e_i is i and 2^{ν_j} is the largest power of 2 in $3^{4j} - 1$.

(viii) *If* $k = 8j$, *then* $H^*\Omega_0^k \mathrm{JSO}(3)$ *is isomorphic to*

$$(F_2[\omega_n]/\omega_n^{2^{\nu_j}}) \otimes F_2[f_{k2^{\nu_j}-1}^2] \otimes F_2[f_{2n-1} \mid 2n \not\equiv \bmod 2^{\nu_j}]$$

as an algebra with degree$(\omega_i) = i$ *and* degree$(f_i) = i$.

Turning to other specific examples, we consider $H_*\Omega_0^{n+k}S^n$. Here of course one must first consider the fibrations giving the EHP sequence. Recall the second James–Hopf invariant $h_2 : \Omega S^{n+1} \to \Omega S^{2n+1}$ together with the j-fold composite of h_2,

$$h_2^j : \Omega S^{n+1} \to \Omega S^{2^j n+1}.$$

The 2-local fibre of h_2^j is $J_{2^j-1}S^n$, the $(2^j - 1)$-st filtration of the James construction JS^n. Thus one obtains a fibration

$$\Omega^{n+k}(h_2^j) : \Omega_0^{n+k+1}S^{n+1} \to \Omega^{n+k+1}S^{2^j n+1}$$

where we assume that $n + k + 1 < 2^j n$ to insure that the base is simply-connected. Thus the fibre of $\Omega^{n+k}(h_2^j)$ is $\Omega_0^{n+k}(J_{2^j-1}S^n)$ and there is a fibre sequence

$$\Omega^{n+k+2}S^{2^j n+1} \xrightarrow{\Omega^{n+k}(\Delta)} \Omega_0^{n+k}(J_{2^j-1}S^n) \longrightarrow \Omega_0^{n+k+1}S^{n+1} \longrightarrow \Omega^{n+k+1}S^{2^j n+1}.$$

LEMMA 1.7. *The Serre spectral sequence in* mod-2 *homology for* $\Omega^{n+k}(h_2^j)$ *collapses if and only if* $\Omega^{n+k}(\Delta)_*$ *is zero on* $H_{2^j n+1-(n+k+2)}(\quad; F_2)$.

Thus the failure of the collapse of the Serre spectral sequence is measured by the mod-2 Hurewicz image of the generalized Whitehead product. We show that this image is frequently non-zero.

THEOREM 1.8. *Let* $n = 2^a(2k+1) - 1$ *with* $k > 0$. *Then the Whitehead square* ω_n *has non-trivial Hurewicz image in* $H_{n-q-1}(\Omega_0^{n+q}S^n; F_2)$ *for* $n - 1 > q \geqq 2^a$.

Thus, for example, if $n = 4j + 1$, then ω_n has non-trivial Hurewicz image in $H_{n-3}(\Omega_0^{n+2}S^n; F_2)$ and by [H] zero image in $H_{n-2}(\Omega_0^{n+1}S^n; F_2)$. Also, if $n = 2j$, then ω_n has non-trivial image in $H_{n-2}(\Omega_0^{n+1}S^n; F_2)$. Of course by desuspending the Whitehead square one obtains other classes in $H_*(\Omega^{n+q-\varepsilon}S^{n-\varepsilon}; F_2)$, but we shall not pursue this connection to the vector field problem here. As a sample specific calculation we illustrate how the Whitehead square forces relations in the homology of $\Omega_0^4 S^3$ and, in general, in $H_*(\Omega_0^{2n}S^{2n-1}; F_2)$ as in [H].

EXAMPLE 1.9. There is an isomorphism of algebras

$$H_*(\Omega_0^4 S^3; F_2) \cong \Lambda[Q_1^a Q_2^b[1] \mid a + b \geqq 1] \otimes F_2[Q_1^a Q_2^b Q_3^c Q_4^d x_1 \mid d + b \geqq 1]$$
$$\otimes F_2[(Q_1^c Q_3^d x_1)^2, x_1^2 \mid c + d > 0].$$

Thus the image of $H_*(\Omega_0^3 S^3; F_2)$ in $H_*(\Omega_0^4 S^3; F_2)$ is the exterior algebra given by $\Lambda[Q_1^a Q_2^b[1] \mid a + b \geqq 1]$.

We remark that the Serre spectral sequence for

$$\Omega^{n+k}(h_2^j) : \Omega_0^{n+k+1} S^{n+1} \to \Omega^{n+k+1} S^{2^j n+1}$$

never collapses for $j \geqq 2$ and $k \geqq 1$ by the following proposition.

PROPOSITION 1.10. *If $k \geqq 1$, $j \geqq 2$, and $n + k + 2 < 2^j n + 1$ then the composite θ,*

$$\begin{array}{ccc} \Pi_{2^j n+1-(n+k+2)} \Omega^{n+k+2} S^{2^j n+1} & \xrightarrow{\Omega^{n+k}(\Delta)_*} & \Pi_{2^j n-n-k} \Omega_0^{n+k} (J_{2^j-1} S^n) \\ & \searrow^{\theta} & \downarrow \\ & & H_{2^j n-n-k}(\Omega_0^{n+k} J_{2^j-1} S^n; F_2), \end{array}$$

is non-zero.

The reason for mentioning Proposition 1.10 is that one can iteratively fibre $\Omega_0^{n+k+1} S^{n+1}$ into spaces which one can eventually identify. These calculations suggest that there should be a "reasonable" spectral sequence abutting to $H_*(\Omega_0^{n+k} S^n; F_2)$ with a computable E^1-term.

2. The homology of $\Omega_0^{i+\infty} S^\infty$, $i = 1, 2$

The methods here are to consider the Eilenberg–Moore spectral sequence abutting to $H^*\Omega X$ with $E_2 = \mathrm{Tor}_{H^*X}(F_2, F_2)$ (and where X is assumed to be simply-connected). Recall that if H^*X is a polynomial algebra and X is of finite type, then the spectral sequence collapses. The work here is to determine the precise algebra extension by using the action of the Steenrod operations.

Let X be an ∞-loop space and $x \in H_q X$. Define three functions ξ, λ, and λ' as follows:

(a) If $q \equiv 1 \bmod 2$, then $\xi(x) = 0$ while if $q = 2k$, then $\xi(x) = Sq_*^k(x)$.
(b) If $q \equiv 0 \bmod 2$, then $\lambda(x) = 0$ and if $q = 2k + 1$, then $\lambda x = Sq_*^k(x)$.
(c) If $q \equiv 1 \bmod 2$, then $\lambda'(x) = 0$ and if $q = 2k$, then $\lambda' x = Sq_*^{k-1}(x)$.

The following lemma is immediate from the Nishida relations:

LEMMA 2.1.

(a)
$$\xi Q_i x = \begin{cases} Q_{i/2}\xi x & \text{if } i \equiv 0 \bmod 2, \\ 0 & \text{if } i \equiv 1 \bmod 2. \end{cases}$$

(b)
$$\lambda Q_i x = \begin{cases} Q_{(i-1)/2}x + Q_{(i+1)/2}\xi x & \text{if } i \equiv 1 \bmod 4, \\ Q_{(i-1)/2}x & \text{if } i \equiv 3 \bmod 4, \\ 0 & \text{if } i \equiv 0 \bmod 2. \end{cases}$$

(c)
$$\lambda' Q_i x = \begin{cases} (i/2-1,2)Q_{i/2+1}\xi x + (i/2,1)Q_{i/2}\lambda x + Q_{i/2-1}\lambda' x & \text{if } i \equiv 0 \bmod 2, \\ 0 & \text{if } i \equiv 1 \bmod 2. \end{cases}$$

Wellington gave a basis for the module of primitives $PH_*\Omega_0^\infty S^\infty$ as follows [W]: let $I = (i_1, \dots, i_k)$, $k \geqq 1$, be an admissible sequence and so $i_j \leqq i_{j+1}$. I is said to be *even* if all of the i_j are 0 mod 2. I is said to be *odd* if at least one i_j is 1 mod 2. Next, define elements $f_I \in PH_*\Omega_0^\infty S^\infty$ as follows:

(i) If $i_k \equiv 1 \bmod 2$, then $f_I = Q_J f_{i_k}$ where $I = (J, i_k)$ and f_k is the Newton polynomial in the elements $x_k = Q_k[1] * [-2]$.

(ii) Let I equal $(J, i_j, i_{j+1}, \dots, i_k)$ with $(i_{j+1}, \dots, i_k)$ even and $i_j \equiv 1 \bmod 2$. Then $f_I = Q_J Q_{2i_{j+1}-i_j} Q_{2i_{j+2}-i_j} \cdots Q_{2i_k - i_j} f_{i_j}$.

THEOREM 2.2 [Wellington]. *The f_I above are a basis for $PH_*\Omega_0^\infty S^\infty$.*

LEMMA 2.3. *The map $\lambda : PH_*\Omega_0^\infty S^\infty \to PH_*\Omega_0^\infty S^\infty$ is an epimorphism.*

PROOF. Since λ is given in terms of Sq_*^k, it preserves primitives and the Nishida relations give $Sq_*^{k-1} f_{2k-1} = f_k$. Notice that Lemma 2.1 gives

$$\lambda Q_i x = \begin{cases} Q_{(i\cdot 1)/2}\lambda x & \text{if } i \equiv 1 \bmod 2, \\ 0 & \text{if } i \equiv 0 \bmod 2. \end{cases}$$

Consider f_I in case (i) for the definition of f_I. Here

$$\lambda(Q_{2i_1+1}Q_{2i_2+1}\cdots Q_{2i_{k-1}+1}f_{2i_k-1}) = f_I.$$

In case (ii) above for the definition of f_I we have

$$f_I = Q_J Q_{2i_{j+1}-i_j} Q_{2i_{j+2}-i_j} \cdots Q_{2i_k-i_j} f_{i_j}$$

with $i_j - 1 \bmod 2$. Write

$$x = Q_{2i_1+1}\cdots Q_{2i_{j-1}+1} Q_{4i_{j+1}-2i_j+1}\cdots Q_{4i_k-2i_j+1} f_{2i_j-1}.$$

Notice that $\lambda(x)=f_I$ and that x is primitive although it may not be one of Wellington's basis elements. The lemma follows, but we remark that λ, of course, has a kernel.

Wellington proves that $H^*\Omega_0^\infty S^\infty$ is a polynomial algebra with generators given by $(PH_*\Omega_0^\infty S^\infty)^*$. As $\Omega_0^\infty S^\infty$ splits as $RP^\infty \times \Omega_0^\infty S^\infty\langle 1\rangle$, the cohomology of $\Omega_0^\infty S^\infty\langle 1\rangle$ is also polynomial. By the collapse of the Eilenberg–Moore spectral sequence abutting to $H^*\Omega_0^{\infty+1}S^\infty$, we get

$$E_\infty \cong \Lambda[\Sigma^{-1}(PH_*\Omega_0^\infty S^\infty\langle 1\rangle)^*].$$

Thus to show that $H^*\Omega_0^{\infty+1}S^\infty$ is polynomial algebra it suffices to show that the squaring map on the (-1)-line of the Eilenberg–Moore spectral sequence is a monomorphism. As the squaring map is dual to λ, this follows from Lemma 2.3. The above gives that there is a choice of polynomial generators in the image of the cohomology suspension and thus $H^*\Omega_0^{\infty+1}S^\infty$ is a primitively generated polynomial algebra and Theorem 1.1 follows.

Lemma 2.4. *There is a 2-local equivalence* $\Omega_0^{\infty+1}S^\infty \simeq RP^\infty \times \Omega_0^{\infty+1}S^\infty\langle 1\rangle$ *and* $H^*\Omega_0^{\infty+1}S^\infty\langle 1\rangle$ *is a polynomial algebra.*

Proof. There is a multiplicative map $\Omega_0^{\infty+1}S^\infty \to RP^\infty$ giving an isomorphism on Π_1. It suffices to exhibit a map $RP^\infty \to \Omega_0^{\infty+1}S^\infty$ inducing an isomorphism on Π_1. Consider $\eta: S^1 \to \Omega_{(1)}^\infty S^\infty$ representing η and $\omega: RP^\infty \to \Omega_{(1)}^\infty S^\infty$ inducing an isomorphism on Π_1. The standard difference construction using the composition pairing and additive loop structure gives a map $f: S^1 \wedge RP^\infty \to \Omega_{(1)}^\infty S^\infty$ inducing an isomorphism on Π_2. Adjointing f gives the desired map and the lemma follows from Theorem 1.1.

We now mimic the proof of 1.1 to prove 1.2. The Eilenberg–Moore spectral sequence abutting to $H^*\Omega_0^{\infty+2}S^\infty$ with $E_2=\mathrm{Tor}_{H^*\Omega^{\infty+1}S^\infty\langle 1\rangle}(F_2,F_2)$ collapses by Lemma 2.4. We claim that the squaring map is a monomorphism on all elements of even degree. Notice that the squaring map is dual to λ' here. Let $Q_I f_k$ be an element of Wellington's basis. Assuming that $Q_I f_k$ is of even degree, we have $I=(i_1,\dots,i_{k-1})$, $i_1\equiv i_2\equiv\cdots\equiv i_{j-1}\equiv 0 \bmod 2$, and $i_j\equiv 1 \bmod 2$ for some j with $2\leqq j\leqq k$. Thus $Q_{i_j}Q_{i_{j+1}}\cdots Q_{i_{k-1}}f_k=\lambda(y)$ for some y by Lemma 3.2. Thus

$$\lambda'(Q_{2i_1+2}\cdots Q_{2(i_{j-2})+2}Q_{2i_{j-1}}y)=Q_{i_1}\cdots Q_{i_{j-1}}\lambda y=Q_I f_k.$$

Assuming that $Q_I f_k$ is of odd degree, then i_1 is odd and $i_1\equiv i_2\equiv\cdots\equiv i_{j-1}\equiv 1 \bmod 2$ with $i_j\equiv 0 \bmod 2$ for $j\leqq k-1$. Then

$$\lambda'(Q_{2i_1+2}\cdots Q_{2i_{j-1}+2}y)=Q_{i_1}\cdots Q_{i_{j-1}}\lambda' y.$$

But $Q_{i_j}\cdots Q_{i_{k-1}}f_k=\lambda' y$ for some y with $i_j\equiv 0 \bmod 2$.

Now assume that $i_1\equiv i_2\equiv\cdots\equiv i_k\equiv 1 \bmod 2$. Then

$$\lambda'(Q_{2i_1+2}\cdots Q_{2i_{k-1}}f_k)=Q_{i_1}\cdots Q_{i_{k-1}}\lambda' f_k=0$$

and so the dual of $Q_I f_k$ in this case corresponds to exterior generators in $H^*\Omega_0^{\infty+2}S^\infty$.

We remark that this calculation is consistent with the fact that $\Pi_1\Omega_0^{\infty+2}S^\infty\cong Z/8$ and thus the cup square of elements in $H^1(\Omega_0^{\infty+2}S^\infty;F_2)$ is zero.

To carry out further calculations one must compute possible differentials in the Eilenberg–Moore spectral sequence. The differentials arise on divided power elements in

$$\mathrm{Tor}_{F_2[x]/x^{2^k}=0}(F_2,F_2).$$

As a sample, one might consider the universal model for such differentials by considering the loop-space of E, the fibre of

$$K(Z/2,n)\xrightarrow[(i)^{2^k}]{} K(Z/2,n2^k).$$

For example, if $k=1$, one can compute some differentials by a factorization of Sq^n, $n\neq 2^j$, as the study of the Whitehead product in [BP]. For the time being we just include the remark that there are non-zero differentials in the Eilenberg–Moore spectral sequence computing $H^*\Omega_0^{\infty+3}S^\infty$.

3. The homology of $\Omega_0^n(\mathrm{JU}(q))$

Consider $\psi^q-1:\mathrm{BU}\to\mathrm{BU}$ with homotopy theoretic fibre $\mathrm{JU}(q)$. Write $f_{q,n}=\Omega^n(\psi^q-1)$. As ψ^q induces multiplication by q^k on $\Pi_{2k}\mathrm{BU}$, $f_{q,2n}:\Omega_0^{2n}\mathrm{BU}\to\Omega_0^{2n}\mathrm{BU}$ induces multiplication by $q^{n+k}-1$ on $\pi_{2k}\Omega_0^{2n}\mathrm{BU}$. Write $H_*\mathrm{BU}=H_*(\mathrm{BU};Z)$ and recall that the module of primitives $PH_*\mathrm{BU}$ is isomorphic to Z in degrees $2k$ with a generator given by the Newton polynomials in x_{2k} where x_{2k} is a choice of generator for the image of $H_{2k}\mathrm{BU}(1)\to H_{2k}\mathrm{BU}$. Thus

(i) $p_2=x_2$ and

(ii) $p_{2k}=(-1)^{k+1}kx_{2k}+\Sigma_{j=1}^{k-1}(-1)^{j+1}x_{2j}\cdot p_{2k-2j}$.

Since the Hurewicz map $\phi:\Pi_{2k}\mathrm{BU}\to H_{2k}\mathrm{BU}$ is a monomorphism with $\phi(1)=k!(p_{2k})$, we have

(iii) $(f_{q,2n})_*(p_{2k})=(q^{n+k}-1)(p_{2k})$,

(iv) $(f_{q,2n})_*(x_2) = (q^{n+1} - 1)(x_2)$, and

(v) $(f_{q,2n})_*(x_{2k})$

$= ((-1)^{k+1}/k)[(q^{n+k} - 1)(p_{2k}) - (f_{q,2n})_*(\Sigma_{j=1}^{k-1}(-1)^{j+1}x_{2j} \cdot p_{2k-2j})]$.

Next, write $H_*\mathrm{SU} = \Lambda[x_{2k+1}]$ where x_i is of degree and $k \geqq 1$. Since $f_{q,2n+1}$ induces multiplication by $q^{n+k} - 1$ on $\Pi_{2k-1}\Omega_0^{2n+1}\mathrm{BU}$, we have

(vi) $f_{q,2n+1*}(x_{2k-1}) = (q^{n+k} - 1)x_{2k-1}$.

Since $\bar{x}_{2k} = \sigma_*(x_{2k-1})$ where σ_* is the homology suspension,

(vii) $f_{q,2n*}(\bar{x}_{2k}) = (q^{n+k} - 1)\bar{x}_{2k}$.

Thus the next lemma follows.

LEMMA 3.1. *After reducing* mod p, *these formulas hold*:

$$(f_{q,2n})^*(c_k) = \begin{cases} 0 & \text{if } p \mid (q^{n+k} - 1) \\ \text{unit} \cdot c_k & \text{if } p \nmid (q^{n+k} - 1) \end{cases}$$

and

$$(f_{q,2n})_*(\bar{x}_{2k}) = \begin{cases} 0 & \text{if } p \mid (q^{n+k} - 1) \\ \text{unit} \cdot \bar{x}_{2k} & p \nmid (q^{n+k} - 1). \end{cases}$$

Next, consider the map of fibrations

$$\begin{array}{ccccccc} U & \longrightarrow & \mathrm{JU}(q) & \longrightarrow & \mathrm{BU} & \xrightarrow{\psi^q - 1} & \mathrm{BU} \\ \downarrow & & \downarrow & & \downarrow{\scriptstyle \psi^q - 1} & & \downarrow{\scriptstyle 1} \\ U & \longrightarrow & * & \longrightarrow & \mathrm{BU} & \xrightarrow[1]{} & \mathrm{BU} \end{array}$$

Passing to connected covers and looping, we get

$$\begin{array}{ccccc} \Omega_0^{2n+1}\mathrm{BU} & \longrightarrow & \Omega_0^{2n}\mathrm{JU}(q) & \longrightarrow & \Omega_0^{2n}\mathrm{BU} \\ {\scriptstyle 1}\downarrow & & \downarrow & & \downarrow{\scriptstyle f_{q,2n}} \\ \Omega_0^{2n+1}\mathrm{BU} & \longrightarrow & * & \longrightarrow & \Omega_0^{2n}\mathrm{BU} \end{array}$$

As $c_k \in H^{2k}(\Omega_0^{2n}\mathrm{BU}; Z)$ is the transgression of $e_{2k-1} \in H^{2k-1}(\mathrm{SU}; Z)$, naturality of the Serre spectral sequence and Lemma 3.1 gives that e_{2k-1} transgresses to $f_{q,2n}^*(c_k)$ in the Serre spectral sequence for computing $H^*\Omega_0^{2n}\mathrm{JU}(q)$. Reducing mod p one has (a) e_{2k-1} is an infinite cycle if and only if p divides $q^{n+k} - 1$ and (b) e_{2k-1} transgresses to a unit multiple of c_k if and only if p does not divide $q^{n+k} - 1$. Thus we have

$$E_2^{**} \cong \left(\bigotimes_{p \nmid (q^{n+k}-1)} \Lambda[e_{2k-1}] \otimes F_p[c_k] \right) \otimes \left(\bigotimes_{p \mid (q^{n+k}-1)} \Lambda[e_{2k-1}] \otimes F_p[c_k] \right)$$

as a differential algebra with

$$d_{2k-1}(e_{2k-1}) = \text{unit} \cdot c_k$$

provided $p \nmid (q^{n+k}-1)$. Since $\Lambda[e_{2k-1}] \otimes F_p[c_k]$ is acyclic in these cases, one has

$$E_\infty^{**} \cong \left(\bigotimes_{p \mid (q^{n+k}-1)} \Lambda[e_{2k-1}] \otimes F_p[c_k] \right).$$

As the homology of SU is exterior and the homology of BU is polynomial, we have proved the following theorem.

Theorem 3.2. *There is an isomorphism of algebras*

$$H_*(\Omega_0^{2n}\mathrm{JU}(q); F_p) \to \bigotimes_{p \mid (q^{n+k}-1)} \Lambda[f_{2k-1}] \otimes F_p[x_{2k}]$$

where f_{2k-1} *is the unique primitive in the image of* $H_{2k-1}(\Omega_0^{2n+1}\mathrm{BU}; F_p)$ *and* x_{2k} *has non-zero image in* $H_{2k}(\Omega_0^{2n}\mathrm{BU}; F_p)$.

Rephrasing the dimensions above, one has an algebra isomorphism

$$H_*(\Omega_0^{2n}\mathrm{JU}(q); F_p) \cong \bigotimes_{j \geq 1} \Lambda[u_{qj-1-2d}] \otimes F_p[v_{qj-2d}]$$

where $|u_i| = i$, $|v_i| = i$, $q = 2p-2$, and $0 \leq d < p-1$ with $n \equiv d \bmod (p-1)$. Thus the first two non-vanishing groups for $\bar{H}_*(\Omega_0^{2n}\mathrm{JU}(q); F_p)$ are in degrees $q-1-2d$ and $q-2d$.

To compute the homology of $\Omega_0^{2n+1}(\mathrm{JU}(q))$ consider the map of fibre sequences

$$\begin{array}{ccccccc}
\mathrm{SU} & \longrightarrow & * & \longrightarrow & \mathrm{BSU} & \xrightarrow{1} & \mathrm{BSU} \\
\downarrow{\scriptstyle \Omega(\psi^q-1)} & & \downarrow & & \downarrow & & \downarrow{\scriptstyle \psi^q-1} \\
\mathrm{SU} & \longrightarrow & \mathrm{JU}(q) & \longrightarrow & \mathrm{BU} & \xrightarrow[\psi^q-1]{} & \mathrm{BU}
\end{array}$$

to get maps of fibrations

$$\begin{array}{ccccc} \Omega_0^{2n-1}(\mathrm{SU}) & \longrightarrow & * & \longrightarrow & \Omega_0^{2n-1}\mathrm{BU} \\ \downarrow{\scriptstyle f_{q,2n}} & & \downarrow & & \downarrow{\scriptstyle 1} \\ \Omega_0^{2n-1}(\mathrm{SU}) & \longrightarrow & \Omega_0^{2n-1}\mathrm{JU}(q) & \longrightarrow & \Omega_0^{2n-1}\mathrm{BU} \end{array}$$

and

$$\begin{array}{ccccc} \mathrm{BU} & \longrightarrow & * & \longrightarrow & \mathrm{SU} \\ \downarrow{\scriptstyle f_{q,2n}} & & \downarrow & & \downarrow{\scriptstyle 1} \\ \mathrm{BU} & \longrightarrow & \Omega_0^{2n-1}\mathrm{JU}(q) & \longrightarrow & \mathrm{SU} \end{array}$$

Recall that $H_*(\mathrm{SU}; Z)$ is an exterior algebra with generators x_{2k+1}, $k \geqq 1$, which transgress to x_{2k} in $H_{2k}(\mathrm{BU}; Z)$ in the Serre spectral sequence for the path fibration. Thus when considering the Serre spectral sequence for

$$(*) \qquad \mathrm{BU} \to \Omega_0^{2n-1}\mathrm{JU}(q) \to \mathrm{SU}$$

one has that x_{2k+1} transgresses to $(f_{q,2n})_*(x_{2k})$. By reduction mod p together with Lemma 3.1, we have

(i) x_{2k+1} is an infinite cycle if and if $p \mid q^{n+k} - 1$, and

(ii) x_{2k+1} transgresses to unit $\cdot\, x_{2k}$ modulo decomposables if $p \nmid q^{n+k} - 1$.

Thus the Serre spectral sequence in mod p homology has the form

$$E^2_{**} \cong \Bigg(\bigotimes_{p \nmid (q^{n+k}-1)} \Lambda[x_{2k+1}] \otimes F_p[x_{2k}]\Bigg) \otimes \Bigg(\bigotimes_{p \mid (q^{n+k}-1)} \Lambda[x_{2k+1}] \otimes F_p[x_{2k}]\Bigg)$$

with

$$d(x_{2k+1}) = \begin{cases} 0 & \text{if } p \mid (q^{n+k} - 1), \\ \text{unit} \cdot x_{2k} & \text{if } p \nmid (q^{n+k} - 1). \end{cases}$$

Thus one has

$$E^{\infty}_{**} \cong \bigotimes_{p \mid (q^{n+k}-1)} \Lambda[x_{2k+1}] \otimes F_p[x_{2k}].$$

If p is an odd prime, there is no problem in determining the algebra extension. If $p = 2$, it is conceivable that a representative of x_{2k+1} does not have height 2 in the Pontrjagin ring. However, by the above we have that the Eilenberg–Moore spectral sequence with

$$E^2_{**} = \mathrm{Tor}^{H_*\Omega_0^{2n}\mathrm{JU}(q)}(F_2, F_2)$$

abutting to $H_*(\Omega_0^{2n-1}\mathrm{JU}(q); F_2)$ collapses. Thus $(x_{2k+1})^2$ is a multiple of the unique primitive in degree $4k+2$.

Theorems 1.3 and 1.4 follow.

4. The mod 2 cohomology of $\Omega_0^n\mathrm{JSO}(3)$

In this section we prove Theorem 1.6. Throughout this section all coefficients are in F_2 unless otherwise stated. Write $X\langle k\rangle$ for the k-connected cover of X. Recall that JSO(3) is the fibre of $\psi^3 - 1 : \mathrm{BSO} \to \mathrm{BSO}$. The next lemma follows directly from the action of ψ^3 on $\Pi_*\mathrm{BSO}$ together with real Bott periodicity: ψ_*^3 acts by multiplication by 1 on Π_q for $q \equiv 1$ or 2 mod 8 and is multiplication by 3^{2k} on $\Pi_{4k}\mathrm{BSO}$.

LEMMA 4.1. *If $k \geqq 0$ there are fibrations*

(1) $\mathrm{SO}\langle k\rangle \to \mathrm{JSO}(3)\langle k\rangle \to \mathrm{BSO}\langle k\rangle$ *if* $k \not\equiv 3$ mod 4,

and

(2) $\mathrm{SO}\langle k\rangle \to \mathrm{JSO}(3)\langle k\rangle \to \mathrm{BSO}\langle k+1\rangle$ *if* $k \equiv 3$ mod 4.

Next notice that if $k \equiv 0, 1$ mod 8, then the map $i : \mathrm{JSO}(3) \to \mathrm{BSO}$ gives an epimorphism on Π_{k+1}. Since a multiplicative fibration $\Omega\Pi : \Omega E \to \Omega B$ giving an epimorphism on Π_1 has trivial local coefficients, one has the next lemma.

LEMMA 4.2. *There are fibrations with trivial local coefficients given by*

(1) $\Omega_0^k\mathrm{SO} \to \Omega_0^k\mathrm{JSO}(3) \to \Omega_0^k\mathrm{BSO}$ *if* $k \not\equiv 3$ mod 4,

and

(2) $\Omega_0^k\mathrm{SO} \to \Omega_0^k\mathrm{JSO}(3) \to \Omega^k\mathrm{BO}\langle k+1\rangle$ *if* $k \equiv 3$ mod 4.

The calculation of the additive substructure of $H^*(\Omega^k\mathrm{JSO}(3))$ if $k \not\equiv 0, 7$ mod 8 is a formal consequence of Bott periodicity together with some remarks about Hopf algebras. To do this we recall the spaces occurring in real Bott periodicity together with their cohomology. References are [B] and [C].

(0) $\mathrm{BO} \simeq \mathrm{BSO} \times \mathrm{RP}^\infty$,

(1) $\Omega(\mathrm{BSO}) \simeq \mathrm{SO} \simeq \mathrm{Spin} \times \mathrm{RP}^\infty$,

(2) $\Omega(\mathrm{Spin}) \simeq \mathrm{SO}/U$,

(3) $\Omega(\mathrm{SO}/U) \simeq U/\mathrm{Sp} \simeq \mathrm{SU}/\mathrm{Sp} \times S^1$,

(4) $\Omega(\mathrm{SU}/\mathrm{Sp}) \simeq \mathrm{BSp}$,

(5) $\Omega(\mathrm{BSp}) \simeq \mathrm{Sp}$,

(6) $\Omega(\mathrm{Sp}) \simeq \mathrm{Sp}/U$,

(7) $\Omega(\mathrm{Sp}/U) \simeq U/O \simeq \mathrm{SU}/\mathrm{SO} \times S^1$,

(8) $\Omega(\mathrm{SU}/\mathrm{SO}) \simeq \mathrm{BO}$.

The relevant cohomology groups are given in [C]:

(i) $H^*\mathrm{BO} \cong F_2[\omega_i \mid i \geqq 1]$,

(ii) $H_*\mathrm{SO} \cong \Lambda[e_i \mid i \geqq 1]$, $H^*\mathrm{SO} \cong F_2[f_{2i-1} \mid i \geqq 1]$ [17–08],

(iii) $H^*\mathrm{SO}/U \cong F_2[c_{4k+2} \mid k \geqq 0]$ [17–21],

(iv) the integral cohomology of U/Sp is torsion free with $H^*(U/\mathrm{Sp}; Z) \cong \Lambda[a_{4k+1} \mid k \geqq 0]$ with a_{4k+1} primitive [17–07],

(v) the integral cohomology of BSp is torsion free and $H^*(\mathrm{BSp}; Z) \cong Z[p_{4k} \mid k \geqq 1]$ [17–05],

(vi) the integral cohomology of Sp is torsion free and $H^*(\mathrm{Sp}; Z) \cong \Lambda[f_{4k+3} \mid k \geqq 0]$,

(vii) the integral cohomology of Sp/U is torsion free and primitively generated with

$$H^*(\mathrm{Sp}/U; F_2) \cong \Lambda[x_2', x_4', \ldots, x_{2k}', \ldots],$$

$$H_*(\mathrm{Sp}/U; Z) \cong Z[u_{4k+2} \mid k \geqq 0] \quad [17\text{–}09],$$

(viii) $H^*\mathrm{SU}/\mathrm{SO} \cong \Lambda[z_k \mid k \geqq 2]$ [17–24],
$H_*U/O \cong F_2[p_1, p_3, \ldots, p_{2k+1}, \ldots]$ [17–22].

Remark. Throughout the above statements, the subscript of a symbol gives its degree.

Next we record some lemmas implied by the cohomology above.

Lemma 4.3. *Let Ωf: SO → SO be a 2-local equivalence. Then $(\Omega f)_* = 1$.*

Proof. Since $H_*\mathrm{SO} \cong \Lambda[e_i]$ and $(\Omega f)_*$ is a multiplicative isomorphism, $(\Omega f)_*(e_i) = e_i + \Delta_i$ where Δ_i is decomposable. Notice that $(\Omega f)_*(e_1) = e_1$ and we may inductively assume that $(\Omega f)_*(e_i) = e_i$ for $i < N$. A calculation with the coproduct then gives that Δ_N is primitive. Since Δ_N is decomposable, this means that $\Delta_N = 0$ as $\mathrm{PH}_*\mathrm{SO}$ has basis

$$p_{2k+1} = e_{2k+1} + \sum_{0<i<j} e_i e_j.$$

Lemma 4.4. *Let $f: X \to X$ be a map of 1-connected spaces with $f^* = 1$ and H^*X is a polynomial algebra. Then $(\Omega f)^* = 1$.*

Proof. The Eilenberg–Moore spectral sequence with $E_2 = \mathrm{Tor}_{H^*X}(F_2, F_2)$ abutting to $H^*\Omega X$ collapses. A choice of multiplicative generators for $H^*\Omega X$ is in the image of the cohomology suspension. The lemma follows by naturality.

LEMMA 4.5. *Let* $\Omega f: \Omega X \to \Omega X$ *be a 2-local equivalence where* ΩX *is* Sp, Sp/U *or* SU/Sp. *Then* $(\Omega f)_* = 1$.

PROOF. $(\Omega f)^*(a_{4k+1}) = a_{4k+1}$ as a_{4k+1} is primitive. $(\Omega f)^*(f_{4k+3}) = f_{4k+3}$ as f_{4k+3} is primitive. A similar calculation applies to $H^*\mathrm{Sp}/U$. The lemma follows.

LEMMA 4.6. *Let* $(\Omega f): \mathrm{BSp} \to \mathrm{BSp}$ *be a 2-local equivalence. Then* $(\Omega f)^* = 1$ (*in* mod 2 *cohomology*).

PROOF. Consider $\Omega^2 f$: Sp $\to$ Sp. By Lemma 4.5 $(\Omega^2 f)_* = 1$. As $H_*\mathrm{BSp}$ is isomorphic to a polynomial algebra as an algebra, the lemma follows.

We use the above to compute $H^*\Omega_0^n \mathrm{JSO}(3)$ for $n \not\equiv 0$, 7 mod 8.

LEMMA 4.7. *Let* $\Omega f: \Omega X \to \Omega X$ *be a 2-local equivalence with* $(\Omega f)^* = 1$. *Then* $(\Omega f - 1)^* = 0$.

PROOF. Since $(\Omega f)^* = 1$, this follows immediately from the definition of the conjugation in a Hopf algebra.

THEOREM 4.8. (i) *If* $k \not\equiv 3$ mod 4 *and* $k \not\equiv 0$, 7 mod 8, *the cohomology Serre spectal sequence for*

$$\Omega_0^k \mathrm{SO} \to \Omega_0^k \mathrm{JSO}(3) \to \Omega^k \mathrm{BSO}$$

collapses.

(ii) *If* $k \equiv 3$ mod 8, *the cohomology Serre spectral sequence for*

$$\Omega_0^k \mathrm{SO} \to \Omega_0^k \mathrm{JSO}(3) \to \Omega^k \mathrm{BO}\langle k+1 \rangle$$

collapses.

PROOF. Consider the case $k \equiv 1$ mod 8 and the path-space fibration $\Omega\mathrm{Spin} \to * \to \mathrm{Spin}$. Since $H^*\mathrm{Spin} \cong F_2[f_{2i-1} \mid i \geqq 2]$, one has that there is a choice of polynomial generators for $H^*\Omega\mathrm{Spin} = F_2[c_{4k+2}]$ which are in the image of the cohomology suspension (by collapse of the Eilenberg–Moore spectral sequence). Next recall that $\Omega^{k-1}(\psi^3 - 1): \mathrm{BSO} \to \mathrm{BSO}$ factors through BSpin to give a map of fibrations

$$\begin{array}{ccccc} \Omega(\mathrm{Spin}) & \longrightarrow & \Omega_0^k \mathrm{JSO}(3) & \longrightarrow & \mathrm{SO} \\ \| & & \downarrow & & \downarrow{\scriptstyle \Omega(f_k)} \\ \Omega(\mathrm{Spin}) & \longrightarrow & * & \longrightarrow & \mathrm{Spin} \end{array}$$

Thus the Serre spectral sequence for the upper fibration collapses by Lemmas 4.3 and 4.7 as $\Omega^k(\psi^3 - 1)^*$ is trivial. Finally, the algebra extension is trivial as $H^*\Omega\mathrm{Spin}$ is polynomial.

Next consider the case $k \equiv 2 \bmod 8$ and the path space fibration $\Omega(\mathrm{SO}/U) \to * \to \mathrm{SO}/U$. The argument is similar to the above: (1) there is a choice of algebra generators for $H^*\Omega(\mathrm{SO}/U)$ in the image of the cohomology suspension, (2) $\Omega^k(\psi^3 - 1) : \mathrm{SO}/U \to \mathrm{SO}/U$ is trivial in cohomology by Lemmas 4.4 and 4.7 as $\Omega(\mathrm{Spin}) \simeq \mathrm{SO}/U$, and (3) there is a map of fibrations

$$\begin{array}{ccccc}
\Omega(\mathrm{SO}/U) \simeq \Omega_0^k \mathrm{SO} & \longrightarrow & \Omega_0^k \mathrm{JSO}(3) & \longrightarrow & \Omega_0^k \mathrm{BO} \simeq \mathrm{SO}/U \\
\downarrow{\scriptstyle 1} & & \downarrow & & \downarrow{\scriptstyle \Omega^k(\psi^3 - 1)} \\
\Omega_0^{k+1}\mathrm{BO} & \longrightarrow & * & \longrightarrow & \Omega_0^k \mathrm{BO}
\end{array}$$

The next case is $k \equiv 3 \bmod 8$ where we consider the fibration $\Omega_0^k \mathrm{SO} \to \Omega_0^k \mathrm{JSO}(3) \to \Omega_0^k \mathrm{BO}\langle k+1 \rangle$. By Bott periodicity, $\Omega_0^k \mathrm{BO}\langle k+1 \rangle \simeq \mathrm{SU}/\mathrm{Sp}$ and $\Omega_0^k \mathrm{SO} \simeq \mathrm{BSp}$. Thus there is a map of fibration sequences

$$\begin{array}{ccccccc}
\Omega_0^k \mathrm{SO} & \longrightarrow & * & \longrightarrow & \Omega_0^k \mathrm{BO}\langle k+1 \rangle & \xrightarrow{\;1\;} & \Omega_0^k \mathrm{BO}\langle k+1 \rangle \\
{\scriptstyle \Omega^{k+1}(\psi^3 - 1)}\downarrow & & \downarrow & & \downarrow{\scriptstyle 1} & & \downarrow{\scriptstyle \Omega^k(\psi^3 - 1)} \\
\Omega_0^k \mathrm{SO} & \longrightarrow & \Omega_0^k \mathrm{JSO}(3) & \longrightarrow & \Omega_0^k \mathrm{BO}\langle k+1 \rangle & \xrightarrow{\Omega^k(\psi^3 - 1)} & \Omega_0^k \mathrm{BO}\langle k+1 \rangle
\end{array}$$

As $H_*(\mathrm{BSp}; F_2)$ is a polynomial algebra, the Milnor spectral sequence with $E^2 = \mathrm{Tor}^{H_*(\mathrm{BSp}, F_2)}(F_2, F_2)$ abutting to $H_*(\mathrm{SU}/\mathrm{Sp}; F_2)$ collapses. Thus there is a choice of primitives for $H_*(\mathrm{SU}/\mathrm{Sp}; F_2)$ in the image of the homology suspension. As $\Omega^{k+1}(\psi^3 - 1)$ is trivial in cohomology by Lemmas 4.6 and 4.7, the algebra generators in homology of $\Omega_0^k \mathrm{BO}\langle k+1 \rangle$ are infinite cycles. Since all fibrations here are multiplicative, the result follows.

Assume that $k \equiv 4 \bmod 8$ and consider the Serre spectral sequence for the fibration $\mathrm{Sp} \to * \to \mathrm{BSp}$. An inspection shows that the algebra generators in $H^*\mathrm{Sp}$ transgress. (Note that $\mathrm{Ext}_{H_*\mathrm{Sp}}(F_2, F_2)$ collapses and abuts to $H^*\mathrm{BSp}$.) As $\Omega^k(\psi^3 - 1)$ is trivial in (mod 2) cohomology by Lemma 4.6 and 4.7, the collapse follows by comparing the following map of fibrations:

$$\begin{array}{ccccc}
\mathrm{Sp} & \longrightarrow & \Omega_0^k \mathrm{JSO}(3) & \longrightarrow & \mathrm{BSp} \\
\downarrow{\scriptstyle 1} & & \downarrow & & \downarrow{\scriptstyle \Omega^k(\psi^3 - 1)} \\
\mathrm{Sp} & \longrightarrow & * & \longrightarrow & \mathrm{BSp}
\end{array}$$

Assume that $k \equiv 5 \bmod 8$ and consider the fibration $\mathrm{Sp}/U \to \Omega_0^k \mathrm{JSO}(3) \to \mathrm{Sp}$. Next consider the path space fibration with $\mathrm{Cotor}^{H_*(\mathrm{Sp};F_2)}(F_2, F_2)$ abutting to $H_*(\mathrm{Sp}/U; F_2)$. Notice that the spectral sequence collapses and so the exterior generators for $H_*\mathrm{Sp}$ all transgress. Comparing the fibrations

$$\begin{array}{ccccc} \mathrm{Sp}/U & \longrightarrow & * & \longrightarrow & \mathrm{Sp} \\ \downarrow{\scriptstyle \Omega^{k+1}(\psi^3-1)} & & \downarrow & & \downarrow{\scriptstyle 1} \\ \mathrm{Sp}/U & \longrightarrow & \Omega_0^k \mathrm{JSO}(3) & \longrightarrow & \mathrm{Sp} \end{array}$$

we see that the elements in the homology of Sp are infinite cycles in the Serre spectral sequence for the bottom fibration by Lemmas 4.5 and 4.7 together with $\mathrm{Cotor}^{H_*\mathrm{Sp}}(F_2, F_2) \cong H_*(\mathrm{Sp}/U)$.

Assume that $k \equiv 6 \bmod 8$ and consider the map of fibrations

$$\begin{array}{ccccc} \Omega(\mathrm{Sp}/U) & \longrightarrow & \Omega_0^k \mathrm{JSO}(3) & \longrightarrow & \Omega(\mathrm{Sp}) \\ \downarrow{\scriptstyle 1} & & \downarrow & & \downarrow{\scriptstyle \Omega^k(\psi^3-1)} \\ U/O \simeq \Omega(\mathrm{Sp}/U) & \longrightarrow & * & \longrightarrow & \Omega(\mathrm{Sp}) \simeq \mathrm{Sp}/U \end{array}$$

Since $H^*(\mathrm{Sp}/U; F_2)$ is polynomial and H^*U/O is isomorphic to $\mathrm{Tor}_{H^*\mathrm{Sp}/U}(F_2, F_2)$, the algebra generators for H^*U/O are in the image of the cohomology suspension. Since $\Omega^k(\psi^3 - 1)$ is zero in (mod-2) cohomology, the Serre spectral sequence in cohomology collapses for the top fibration.

Finally, we consider the cases $k \equiv 0,\ 7 \bmod 8$. Let $\eta - 1 : \mathrm{BO}(1) \to \mathrm{BO}$ denote the reduced Hopf bundle. Thus $\psi^3(\eta - 1) = \eta^3 - 1 = \eta - 1$ as the square of a line bundle is trivial. Thus there is a homotopy commutative diagram

$$\begin{array}{ccc} & \mathrm{Rp}^\infty & \\ {\scriptstyle \eta-1}\swarrow & & \searrow{\scriptstyle \eta-1} \\ \mathrm{BO} & \xrightarrow[\psi^3]{} & \mathrm{BO} \end{array}$$

and so $\psi^3_* = 1$ as $H_*\mathrm{BO}$ is the symmetric algebra on $(\eta - 1)_* \bar{H}_* \mathrm{RP}^\infty$. [Recall that $\psi^3(\alpha + \beta) = \psi^3\alpha + \psi^3\beta$.] Thus as in [FP] the Serre spectral sequence for $\mathrm{SO} \to \mathrm{JSO}(3) \to \mathrm{BSO}$ collapses.

Write $B : \mathrm{BO} \to \Omega_0^{8k}\mathrm{BO}$ for the Bott equivalence and let $f : X \to \mathrm{BO}$ specify a stable bundle over the finite complex X and let $\beta_k : S^{8k} \to \mathrm{BO}$ denote a generator of $\Pi_{8k}\mathrm{BO} \cong Z$. The isomorphism

$$B_*: [X, \mathrm{BO}] \xrightarrow{\cong} [S^{8k} \wedge X, \mathrm{BO}]$$

is specified by sending f to $\beta_k \otimes f$. Compute the composite $\theta(f)$ given by

$$X \xrightarrow{f} \mathrm{BO} \xrightarrow{B} \Omega_0^{8k}\mathrm{BO} \xrightarrow{\Omega^{8k}(\psi^3-1)} \Omega_0^{8k}\mathrm{BO} \xrightarrow{B^{-1}} \mathrm{BO}$$

as follows (where $\theta = B^{-1} \circ \Omega^{8k}(\psi^3 - 1) \circ B$):

(i) $B(f) = \alpha \otimes f$,

(ii) $\Omega^{8k}(\psi^3 - 1)(B(f)) = (\psi^3 - 1)(\alpha \otimes f) = (\psi^3\alpha \otimes \psi^3 f) - (\alpha \otimes f)$, and

(iii) $\theta(f) = B^{-1}[\psi^3\alpha \otimes \psi^3 f - (\alpha \otimes f)]$
$= B^{-1}[3^{4k}(\alpha \otimes \psi^3 f) - (\alpha \otimes f)]$
$= 3^{4k}(\psi^3 f) - f.$

Since $\psi^3_* = 1$, $\theta^* = (3^{4k} - 1)^*$. Finally compute θ^* on the total Stiefel-Whitney class $W = \Sigma_{i \geq 0} w_i$: In cohomology θ has the same effect as $(3^{4k} - 1)^*$ because $(\psi^3)^* = 1$, and so

$$\theta^*(W) = (W)^{3^{4k}-1}.$$

Thus if $k = 0$, θ^* is trivial. But if $k > 0$, θ^* is non-trivial. Write ν_k for the 2-adic valuation of $3^{4k} - 1$, the largest power of 2 dividing $3^{4k} - 1$.

LEMMA 4.9. *$\theta^*(W) = \Sigma_{i \geq 0}(w_i + \lambda_i)^{2^{\nu_k}}$ where λ_i is in the subalgebra of $H^*(\mathrm{BO}; F_2)$ generated by $w_1, \ldots, w_{i-1}$.*

PROOF. $3^{4k-1} = 2^{\nu_k} \cdot \text{odd number} = 2^{\nu_k} \cdot q$. Thus

$$\theta^*(W) = W^{2^{\nu_k} \cdot q} = (W^{2^{\nu_k}})^q = \left(\sum_{i \geq 0} w_i^{2^{\nu_k}}\right)^q.$$

Since q is odd, it follows that

$$\theta^*(W) = (\Sigma(w_i^{2^{\nu_k}})^{2L})(\Sigma w_i^{2^{\nu_k}}) = (\Sigma w_i^{2^{\nu_k+1}})^L(\Sigma w_i^{2^{\nu_k}})$$

where $q = 2L + 1$ and the lemma follows.

The lemma implies the following corollary.

COROLLARY 4.10.

$$\theta^*(w_n) = \begin{cases} 0 & \text{if } n \not\equiv 0 \bmod 2^{\nu_k} \\ w_j^{2^{\nu_k}} + \lambda_j^{2^{\nu_k}} & \text{if } n = j2^{\nu_k} \end{cases}$$

where λ_i is in the subalgebra generated by $w_1^{2^{\nu_k}}, \ldots, w_{j-1}^{2^{\nu_k}}$.

Next consider the Serre spectral sequence for

$$\mathrm{SO} \to \Omega_0^{8k}\mathrm{JSO}(3) \to \mathrm{BO}.$$

Since $\psi^3 - 1$ factors through BSO, we get

$$\begin{array}{ccccc} \mathrm{SO} & \longrightarrow & \Omega_0^{8k}\mathrm{JSO}(3) & \longrightarrow & \mathrm{BO} \\ \| & & \downarrow & & \downarrow{\scriptstyle \Omega^{8k}(f_k)} \\ \mathrm{SO} & \longrightarrow & * & \longrightarrow & \mathrm{BSO} \end{array}$$

Hence the Serre spectral sequence for the upper fibration is as follows:

$$E_2 \cong F_2[w_n] \otimes F_2[f_{2n-1}], \qquad n \geqq 1.$$

As a differential coalgebra

$$E_2 \cong (F_2[w_i^{2^{\nu_k}}] \otimes F_2[w_i]/(w_i^{2^{\nu_k}} = 0)) \otimes (F_2[f_{j2^{\nu_k}-1}] \otimes F_2[f_{2n-1} \mid 2n \not\equiv 0 \bmod 2^{\nu_k}]).$$

By the above $d_{j2^{\nu_k}}(f_{j2^{\nu_k}-1})$ is the class of $w_j^{2^{\nu_k}}$ as $w_i^{2^{\nu_k}}$, $i < j$, has been killed earlier. Since $\mathrm{Sq}^{j2^{\nu_k-1}}(W_j^{2^{\nu_k}} + \lambda_j^{2^{\nu_k}}) = 0$, it follows that $(f_{j2^{\nu_k}-1})^2$ is an infinite cycle. Since the differential coalgebra $\Lambda[x] \otimes F_2[dx]$ is acyclic, we have

$$E_\infty \cong (F_2[w_n]/w_n^{2^{\nu_k}} = 0) \otimes F_2[f_{j2^{\nu_k}-1}^2] \otimes F_2[f_{2n-1} \mid 2n \not\equiv 0 \bmod 2^{\nu_k}]$$

as an algebra. Since $H^*(\mathrm{SO}; F_2)$ is polynomial, the algebra extension is trivial and

$$H^*\Omega_0^{8k}\mathrm{JSO}(3) \cong F_2[w_n]/w_n^{2^{\nu_k}} = 0 \otimes F_2[f_{j2^{\nu_k}-1}^2] \otimes F_2[f_{2n-1} \mid 2n \not\equiv 0 \bmod 2^{\nu_k}].$$

The next case is $k \equiv 7 \bmod 8$. Consider $\mathrm{Tor}^{H_*\mathrm{BO}}(F_2, F_2) \cong \Lambda[(-1, e_n)]$, $n \geqq 1$. Thus $\mathrm{Tor}^{H_*\mathrm{BO}}(F_2, F_2) \cong E^0 H_* \mathrm{SU/SO}$ and so the homology suspension, $\sigma : \mathrm{QH}_*\mathrm{BO} \to \mathrm{PH}_{*+1}\mathrm{SU/SO}$, is an isomorphism (since $H^*(\mathrm{SU/SO}; F_2) \cong \Lambda[z_k \mid k \geqq 2]$, there is exactly one primitive in $H_q\mathrm{SU/SO}$ for $q \geqq 2$). Thus there is an exact sequence

$$0 \to \mathrm{PH}_{2j+1}\mathrm{SU/SO} \to \mathrm{QH}_{2j+1}\mathrm{SU/SO} \to 0$$

and $0 \to \mathrm{PH}_2\mathrm{SU/SO} \to \mathrm{QH}_2\mathrm{SU/SO} \to 0$. Thus comparing the fibrations

$$\begin{array}{ccccc} \mathrm{BO} & \longrightarrow & * & \longrightarrow & \mathrm{SU/SO} \\ {\scriptstyle \Omega^{8j}(\psi^3-1)}\downarrow & & \downarrow & & \| \\ \mathrm{BO} & \longrightarrow & \Omega_0^{k}\mathrm{JSO}(3) & \longrightarrow & \mathrm{SU/SO} \end{array}$$

we see that p_{2k+1} transgresses to $\Omega^{8j}(\psi^3 - 1)_*(e_{2k})$ where $H_*\mathrm{BO} \cong$

$F_2[e_1, \dots, e_j, \dots]$. Now recall $H_*\mathrm{BO} \cong F_2[\bar{H}_*\mathrm{RP}^\infty]$ with $\eta - 1 : \mathrm{RP}^\infty \to \mathrm{BO}$ the reduced Hopf bundle. By the previous calculation

$$B^{-1} \cdot \Omega^{8j}(\psi^3 - 1) \cdot B(\eta - 1) = (3^{4j} - 1)(\eta - 1).$$

Thus we consider

$$\mathrm{RP}^\infty \to \mathrm{BO} \xrightarrow{3^{4j}-1} \mathrm{BO}$$

in homology where $H_*\mathrm{BO} \cong S[\bar{H}_*\mathrm{RP}^\infty]$ $(= S[(\eta - 1)_*\bar{H}_*\mathrm{RP}^\infty])$ with e_n the generator of the image of $(\eta - 1)_* : H_n\mathrm{RP}^\infty \xrightarrow{(\eta-1)_*} H_n\mathrm{BO}$.

LEMMA 4.11.

$$2^q_*(e_n) = \begin{cases} 0 & \text{if } n \not\equiv 0 \bmod 2^q, \\ [e_{n/2^q}]^{2^q} & \text{if } n \equiv 0 \bmod 2^q. \end{cases}$$

PROOF. If $q = 1$, it is clear and the other cases follow by induction on q.

COROLLARY 4.12.

$$(3^{4j} - 1)_*(e_n) = \begin{cases} 0 & \text{if } n \not\equiv 0 \bmod 2^{\nu_j} \\ [e_{n/2^{\nu_j}}]^{\nu^j} + (\gamma_n)^{\nu^j} & \text{if } n \equiv 0 \bmod 2^{\nu_j} \end{cases}$$

where γ_n is in the ideal generated by e_i, $0 < i < n$.

PROOF. Write $3^{4j} - 1 = 2^{\nu_j} \cdot (2l + 1)$. Thus $3^{4j} - 1$ is the composite

$$\mathrm{BO} \xrightarrow{\Delta} (\mathrm{BO})^{2l+1} \xrightarrow{(2^{\nu_j})^{2l+1}} (\mathrm{BO})^{2l+1} \xrightarrow{\text{multiply}} \mathrm{BO}.$$

$\Delta_*(e_n) = \Sigma e_{i_1} \otimes \cdots \otimes e_{i_{2l+1}}$ and so

$$(3^{4j} - 1)_*(e_n) = \Sigma 2^{\nu_j}(e_{i_1}) \cdots 2^{\nu_j}(e_{i_{2l+1}}).$$

Thus

$$(3^{4j} - 1)_*(e_n) = [e_{n/2^{\nu_j}}]2^{\nu_j} + (\gamma_n)^{2^{\nu_j}}$$

as claimed by Corollary 4.12.

By the above Corollary 4.12, it follows that

(1) $H_*\mathrm{SU/SO} \cong F_2[p_2, p_3, \dots, p_{2k+1}, \dots]$ with $(p_j)^{2^t}$ transgressing to e_{j2^t-1} in the Serre spectral sequence for $\mathrm{BO} \to * \to \mathrm{SU/SO}$,

(2) p_{2k+1} is an infinite cycle in the Serre spectral sequence for $\mathrm{BO} \to \Omega_0^k \mathrm{JSO}(3) \to \mathrm{SU/SO}$ where $k = 8j - 1$ provided $2k \not\equiv 0 \bmod 2^{\nu_j}$,

(3) $(p_j)^2$ is an infinite-cycle in the Serre spectral sequence for $\mathrm{BO} \to \Omega_0^k \mathrm{JSO}(3) \to \mathrm{SU/SO}$,

(4) p_{2k+1} transgress to $[e_{2k/2^{\nu_j}}]^{2^{\nu_j}} + (\text{others})^{2^{\nu_j}}$.

Now consider $E^2 = H_* \mathrm{SU/SO} \otimes H_* \mathrm{BO}$ which we write as a coalgebra as follows:

$$E^2 = \Lambda[p_{2k+1} \mid k \equiv 0 \bmod 2^{\nu_j}] \otimes F_2[p_{2k+1}^2 \mid k \equiv 0 \bmod 2^{\nu_j}]$$
$$\otimes F_2[p_2, p_{2k+1} \mid 2k \not\equiv 0 \bmod 2^{\nu_j}] \otimes F_2[e_n]$$

with $d^{2k+1}(p_{2k+1}) = [e_{2k/2^{\nu_j}}]^{2^{\nu_j}}$ as $[e_i]^{2^{\nu_j}}$, $i < 2^k/2^{\nu_j}$ has been killed earlier. Hence

$$E^\infty \cong F_2[p_{2k+1}^2 \mid k \equiv 0 \bmod 2^{\nu_j}]$$
$$\otimes F_2[p_2, p_{2k+1} \mid 2k \not\equiv 0 \bmod 2^{\nu_j}] \otimes F_2[e_n]/e_n^{2^{\nu_j}} = 0.$$

Since $H_* \mathrm{BO}$ and $H_* \mathrm{SU/SO}$ are both polynomial, there is no extension problem and

$$H_* \Omega_0^k \mathrm{JSO}(3) \cong F_2[p_{2k+1}^2 \mid k \equiv 0 \bmod 2^{\nu_j}]$$
$$\otimes F_2[p_2, p_{2k+1} \mid 2k \not\equiv 0 \bmod 2^{\nu_j}] \otimes F_2[e_n]/e_n^{2^{\nu_j}} = 0.$$

as an algebra where $k = 8j - 1$.

5. "Long" Steenrod operations and spherical homology classes

In this section we record some observations giving some non-trivial elements in the Hurewicz image for certain spaces. We apply this to the homology of $\Omega_0^{n+k} S^n$ and $\Omega_0^k \mathrm{SU}(n)$.

The input is:

(1) a commutative diagram

$$\begin{array}{ccc} E & \longrightarrow & X \\ {\scriptstyle \pi}\downarrow & & \downarrow{\scriptstyle \phi} \\ B & \xrightarrow{\;f\;} & K(G, n) \end{array}$$

(2) a map $\alpha : \Sigma^{n+1} A \to B$ with $f\alpha$ giving a non-zero map on $H_n(\;\;; F_2)$,

(3) B is $(k+1)$-connected with $n > k+1$,

(4) the fibre of π is F and the fibre of ϕ is Y.

LEMMA 5.1. *If $\Omega^k(\phi)$ is null-homotopic, then there is a homotopy commutative diagram*

$$\begin{array}{ccccc}
\Omega^{k+1}\Sigma^{k+1}A & \xrightarrow{\Omega^{k+1}(\alpha)} & \Omega^{k+1}B & \longrightarrow & K(G, n-k-1) \\
& \searrow & \downarrow & & \uparrow \\
& & \Omega_0^k F & \longrightarrow & \Omega_0^k Y
\end{array}$$

and $\Omega^{k+1}(\alpha f)_*$ *factors through* $H_*\Omega_0^k F$.

Proof. Since $\Omega^k(\phi)$ is null-homotopic, $\Omega_0^k Y$ splits as $\Omega_0^k X \times K(G, n-k-1)$. The result follows.

We apply Lemma 5.1 by considering the fibration giving the EHP sequence

$$S^n \longrightarrow \Omega S^{n+1} \xrightarrow{h_2} \Omega S^{2n+1}.$$

Recall that there is a commutative diagram

$$\begin{array}{ccc}
\mathrm{RP}^{n-1} & \longrightarrow & \Omega_0^n S^n \\
\downarrow & & \downarrow \\
\mathrm{RP}^n & \longrightarrow & \Omega_0^{n+1} S^{n+1} \\
\downarrow & & \downarrow \\
S^n & \longrightarrow & \Omega^{n+1} S^{2n+1}
\end{array}$$

Next write $n = 2^a(2k+1) - 1$ for $k \geqq 0$. Inspecting the cohomology of RP^n we get a homotopy commutative diagram

$$\begin{array}{ccc}
\Omega_0^n S^n & \longrightarrow & E \\
\downarrow & & \downarrow \\
\Omega_0^{n+1} S^{n+1} & \longrightarrow & K(Z_2, 2^a k + 2^a - 1) \\
\downarrow & & \downarrow \mathrm{Sq}^{k2^a} \\
\Omega^{n+1} S^{2n+1} & \xrightarrow{f} & K(Z_2, 2^{a+1} k + 2^a - 1)
\end{array}$$

where f induces an isomorphism on $H^n(\ ; F_2)$. Since $\Omega^q(\mathrm{Sq}^{k2^a})^*$ is trivial on $H^{2^a k + 2^a - 1 - q}K(Z_2, 2^a k + 2^a - 1 - q)$ for $q \geqq 2^a$, we have proved Theorem 1.8.

Corollary 5.2. *The* mod-2 *reduction of the Hurewicz image of the adjoint for* ω_n *in* $H_{n-q-1}(\Omega_0^{n+q} S^n)$ *is non-zero if* $n = 2^a(2k+1) - 1$, $k > 0$, *and* $n - 1 > q \geqq 2^a$.

EXAMPLE 5.3. If $n = 2k$, Corollary 5.2 gives that the adjoint of ω_n has non-trivial Hurewicz image in $H_{n-2}\Omega^{n+1}S^n$. If $n = 4k + 1$, then the adjoint of the Whitehead square has non-trivial image in $H_{n-3}\Omega_0^{n+2}S^n$. By [H], this is best possible in case $n = 4j + 1$.

Next consider $H^*\mathrm{CP}^n = F_2[x]/x^{n+1} = 0$. Write $n = 2^a(2k+1) - 1$ for $k \geqq 0$. Since $\mathrm{Sq}^{2(2^a k)} x^{2^a k + 2^a - 1} = x^{2^{a+1}k + 2^a - 1} = x^n$, there is a commutative diagram

$$\begin{array}{ccccc}
\Sigma\mathrm{CP}^n & \longrightarrow & \mathrm{SU}(n+1) & \longrightarrow & K(Z_2, 2^{a+1}k + 2^{a+1} - 1) \\
\downarrow & & \downarrow & & \downarrow \mathrm{Sq}^{2^{a+1}k} \\
S^{2n+1} & \longrightarrow & S^{2n+1} & \longrightarrow & K(Z_2, 2n+1)
\end{array}$$

Thus if $q > 2^{a+1} - 1$, $\Omega^q(\mathrm{Sq}^{2^{a+1}k})$ is trivial on

$$H^{2^{a+1}k + 2^{a+1} - 1 - q} K(Z_2, 2^{a+1}k + 2^{a+1} - 1 - q).$$

The following now is a direct consequence of Lemma 5.1.

COROLLARY 5.4. *If* $n = 2^a(2k+1) - 1$ *and* $q > 2^{a+1} - 1$, *then the composite*

$$S^{2n-q} \xrightarrow{E^{q+1}} \Omega^{q+1}S^{2n+1} \xrightarrow{\Omega^{q+1}(\partial)} \Omega_0^q \mathrm{SU}(n)$$

is non-zero on H_{2n-q}.

REMARK. If $a = 0$ or 1, this is best possible by the thesis of D. Waggoner [Wa]. In particular if $a = 0$ or 1 and $q \leqq 2^{a+1} - 1$, then the resulting map in homology is zero. If $a = 0$ or 1 and $q > 2^{a+1} - 1$, the map is non-zero in homology.

Somewhat more generally, one might consider the fibration

$$J_{2^t-1}S^n \longrightarrow \Omega S^{n+1} \xrightarrow{h_2^t} \Omega S^{2^t n+1}$$

and ask whether the map $\Omega^k(h_2^t)$ is an epimorphism in homology.

LEMMA 5.5. *The map* $\Omega^k(h_2^t)$ *induces an epimorphism in homology for* $k \leqq 2^t n - 2$ *if and only if the map*

$$\Pi_{2^t n-k-1}\Omega^{k+2}S^{2^t n+1} \cong H_{2^t n-k-1}\Omega^{k+2}S^{2^t n+1} \rightarrow H_{2^t n-k-1}(\Omega_0^k J_{2^t-1}S^n; F_2)$$

is zero.

PROOF. Consider the map of fibrations

$$\begin{array}{ccccc} \Omega^{k+2}S^{2^l n+1} & \longrightarrow & * & \longrightarrow & \Omega^{k+1}S^{2^l n+1} \\ \downarrow & & \downarrow & & \downarrow{\scriptstyle \Omega^k(\Delta)} \\ \Omega_0^k J_{2^l-1}S^n & \longrightarrow & \Omega_0^{k+1}S^{n+1} & \longrightarrow & \Omega^{k+1}S^{2^l n+1} \end{array}$$

Notice that $\Omega^k(\Delta)_*$ is zero on $H_{2^l n-k-1}$ if and only if $\Omega^k(h_2^l)_*$ is onto $H_{2^l n-k}$. We claim that $\Omega^k(h_2^l)_*$ is onto $H_{2^l n-k}$ if and only if it is onto all of $H_*\Omega^{k+1}S^{2^l n+1}$.

It suffices to show that if $\Omega^k(h_2^l)_*$ is onto $H_{2^l n-k}$, then it is onto. Since $\Omega^k(h_2^l)_*$ commutes with Q_i, $0 \leqq i \leqq k-1$, it suffices to show that there exist elements x_l with $\Omega^k(h_2^l)_*(x_l) = Q_k^l(i_{2^l n-k})$ + others.

By assumption x_0 exists. We claim that setting $x_l = Q_k^l(x_0)$ suffices. Consider the $(k-1)$-fold homology suspension $\sigma^{k-1}: H_q\Omega^k X \to H_{q+k-1}\Omega X$ and $\sigma^{k-1}Q_k^l(x_0) = [\sigma^{k-1}(x_0)]^{2^l}$. Thus $\sigma^{k-1}Q_k^l(x_0)$ has image $(\hat{\imath})^{2^{l+l}}$ in $H_{n2^{l+l}}(\Omega S^{n+1})$ where $\hat{\imath}$ is the fundamental class of $H_n(\Omega S^{n+1})$. Since $h_{2*}^l(\hat{\imath}^{2^{l+l}}) = (i)^{2^l}$ where i is the fundamental class of $H_{n2^l}(\Omega S^{n2^l+1})$ it follows that

$$\Omega^k(h_2^l)_*(x_l) = Q_k^l(i_{2^l n-k}) + \Delta$$

where $\sigma^{k-1}(\Delta) = 0$. The lemma follows.

Next, consider $h_2^l : \Omega S^{n+1} \to \Omega S^{2^l n+1}$ and recall that $(h_2^l)_*(i^{2^l}) = \hat{\imath}$ where $\hat{\imath}$ is the fundamental class of $H_{2^l n}(\Omega S^{2^l n+1})$. Observe that if $n \geqq 2$ and $n = 2j$, then $\mathrm{Sq}_*^{j(2^l-1)}Q_n^l[1] = Q_j^l[1]$. Hence there is a commutative diagram

$$\begin{array}{ccc} \Omega_0^{n+1}S^{n+1} & \longrightarrow & K(Z_2, 2^l n - n - j(2^l-1)) \\ \downarrow & & \downarrow{\scriptstyle \mathrm{Sq}^{j(2^l-1)}} \\ \Omega^{n+1}S^{2^l n+1} & \longrightarrow & K(Z_2, 2^l n - n) \end{array}$$

Notice that $\Omega(\mathrm{Sq}^{j(2^l-1)})$ is trivial in the right-hand fibration and thus we have proved the following result.

Corollary 5.6. *The composite*

$$S^{2^l n-n} \longrightarrow \Omega^{n+1}S^{2^l n+1} \xrightarrow{\Omega^n(\partial)} \Omega_0^{n+1}J_{2^l-1}S^n$$

is non-zero in homology if $n \equiv 0 \bmod 2$.

A similar calculation applies if $n \equiv 1 \bmod 2$; we omit the details.

As a final example, consider the fibration $S^2 \to \Omega S^3 \to \Omega S^5$ giving $S^3 \to \Omega S^3\langle 3\rangle \to \Omega S^5$. A direct calculation applied to $\Omega_0^3 S^3 \to \Omega_0^4 S^3 \to \Omega^4 S^5\langle 5\rangle$ gives the following result.

THEOREM 5.7. *There is an isomorphism of algebras*

$$H_*\Omega_0^4S^3 \cong \Lambda[Q_1^aQ_2^b[1] * [2^{-a-b}] \mid a+b \geqq 1] \otimes F_2[Q_1^aQ_2^bQ_3^cQ_4^dx_1 \mid b+d>0]$$

$$\otimes F_2[Q_1^aQ_3^cx_1, x_1^2 \mid a+c>0].$$

PROOF. By the above $Q_1^aQ_3^cx_1$ transgresses to $(Q_1^aQ^{b2}[1] * [2^{-a-b}])^2 + (m)^2$ where m is in the ideal generated by $(Q_1^{a'}Q_2^{c'}[1] * [2^{-a'-c'}])^2$ of degree strictly less than $(Q_1^aQ_2^c[1])^2$.

6. Proof of Theorem 1.5

Let X be a space and assume that there is a map

$$\Pi : X \to \mathrm{BGL}(F_q)^+ \quad (= \mathrm{Im}\, J \text{ at } p)$$

where p is an odd prime and q is as given in Section 1.

THEOREM 1.5. *Assume that* Π *induces a split epimorphism on the* p-*primary component of* $\Pi_{2p-3}(\mathrm{Im}\, J) \cong Z/p$. *Then* $H_*(\Omega_0^nX; F_p)$ *contains a primitively generated Hopf algebra which is polynomial on infinitely many generators if* $n \geqq 2$.

The proof of Theorem 1.5 depends on the existence of a single primitive element of infinite height in $H_*(\Omega_0^nX; F_p)$.

EXAMPLE 6.1. Give $\Omega_{(1)}^nS^n$, the component of the degree 1 maps in Ω^nS^n, the structure of an H-space by composition of maps. Thus the stabilization map gives a map

$$\Pi : \Omega_{(1)}^nS^n \to \mathrm{Im}\, J$$

which satisfies the hypotheses of Theorem 1.5 if $n \geqq 3$.

LEMMA 6.2. *Let* $\Pi : X \to \mathrm{Im}\, J$ *be a map which gives a split epimorphism on* $\Pi_{2p-3}\mathrm{Im}\, J_{(p)} \cong Z/p$. *Then* Π *induces a split epimorphism on* mod-p *homotopy (but not necessarily integrally even after localization at* p).

PROOF. Consider $\alpha : P^{2p-3}(p) \to X$ and $\bar{\alpha} : P^{2p-2}(p) \to X$ with the first Bockstein of $\bar{\alpha}$ given by α, and $\Pi\alpha$ represents a generator of $\Pi_{2p-3}(\mathrm{Im}\, J; F_p)$. Consider the Adams map $A : P^{N+2p-2}(p) \to P^N(p)$ where $N \geqq 3$. Consider the homotopy commutative diagram

$$\begin{array}{ccccc} & & X & & \\ & \overset{\hat{\alpha}}{\nearrow} & & \overset{\pi}{\searrow} & \\ P^{2p-2}(p) & & & & \operatorname{Im} J \\ & \underset{d}{\searrow} & & \underset{\pi}{\nearrow} & \\ & & \Omega^{\infty}_{(1)}S^{\infty} & & \end{array}$$

where $\hat{\alpha}$ generates $\Pi_{2p-2}(\Omega^{\infty}_{(1)}S^{\infty}; F_p)$. Thus $\Pi \cdot \hat{\alpha} \cdot A^k, \Pi \cdot \alpha, \beta(\Pi \cdot \hat{\alpha} \cdot A^k)$ generate $\Pi_*(\operatorname{Im} J; F_p)$ where β denotes the first Bockstein in mod-p homotopy. The lemma follows.

Notice that the natural map $\Pi : \Omega^3_{(1)}S^3 \to \operatorname{Im} J$ induces a split epimorphism on $\Pi_*(\ ; F_p)$ but is not even onto $\Pi_* \operatorname{Im} J \otimes Z_p$ as the p-torsion in $\Pi_* S^3$ is bounded by p [S].

Lemma 6.3. *Consider $\Omega^n_0 X$ where X is an H-space. If $H_*(\Omega^n_0 X; F_p)$ contains a primitive element of degree non-zero* mod-p, *of infinite height, and $n \geqq 2$, then $H_*(\Omega^n_0 X; F_p)$ contains a primitively generated sub-Hopf algebra which is polynomial on infinitely many generators.*

Proof. Let $x \in H_{2j}(\Omega^n_0 X; F_p)$ where x is primitive and of infinite height. Consider

$$Q_{I_k}(x) = \underbrace{Q_{2p-2} \cdots Q_{2p-2}}_{\longleftarrow k \longrightarrow}(x)$$

which is defined as long as $n \geqq 2$ because $\Omega^n_0 X$ is a retract of $\Omega^{n+1}_0 \Sigma X$. Notice that

$$Q_{I_k}(x) = cQ^{sp^k}Q^{sp^{k-1}} \cdots Q^{sp}Q^s x, \qquad c \neq 0,$$

for $s = j+1$. Also $P^1_* Q_{I_k}(x) = -(1)(1, sp^{k-1}(p-1)-p)Q^{sp^{k-1}}Q_{I_{k-1}}(x)$ and

$$(1, sp^{k-1}(p-1)-p) \equiv \begin{cases} 1 & \text{mod } p \text{ if } k \geqq 2, \\ 1-s & \text{mod } p \text{ if } k = 1. \end{cases}$$

Since $j \not\equiv 0$ mod p by hypothesis, we have the equation

$$(*) \qquad P^{p^k}_* P^{p^{k-1}}_* \cdots P^p_* P^1_* Q_{I_k}(x) = d(x^{p^{k+1}}), \qquad d \neq 0.$$

Let B denote the Hopf algebra which is polynomial on primitive generators $Q_{I_k}(x)$ and define a map of Hopf algebras

$$\theta : B \to H_*(\Omega_0^n X; F_p)$$

by $\theta(Q_{I_k}(x)) = Q_{I_k}(x)$. A basis for the module of primitives in B is given by $(Q_{I_k}(x))^{p^j}$ and there is at most one of these in any fixed degree. Since x has infinite height in $H_*(\Omega_0^n X; F_p)$, equation (*) guarantees that θ is a monomorphism on the module of primitives. Thus θ is a monomorphism and the lemma follows.

Proof of Theorem 1.5. First consider $\Omega^{2n}\Pi : \Omega_0^{2n} X \to \Omega_0^{2n} \operatorname{Im} J$ and notice that the first two non-vanishing mod-p homotopy groups of $\Omega_0^{2n} \operatorname{Im} J$ are in dimensions $q - 2d$ and $q - 2d - 1$ where $q = 2p - 2$, $0 \leqq d < p - 1$ and $n \equiv d \bmod (p-1)$. Thus we get $\lambda : P^{q-2d}(p) \to \Omega_0^{2n} X$ such that $\Omega^{2n}\Pi(\lambda)$ generates $\Pi_{q-2d}(\Omega_0^{2n} \operatorname{Im} J; F_p)$ by Lemma 6.2. But by inspection, the mod-p Hurewicz map

$$\Phi : \Pi_{q-2d}(\Omega_0^{2n} \operatorname{Im} J; F_p) \to H_{q-2d}(\Omega_0^{2n} \operatorname{Im} J; F_p)$$

is an isomorphism. Thus there is a primitive element in $H_{q-2d}(\Omega_0^{2n} X; F_p)$ of infinite height by Theorem 1.3. Since $0 \leqq d < p - 1$, $q - 2d$ is prime to p and Lemma 6.3 applies to give the theorem.

Next consider $\Omega^{2n+1}\Pi : \Omega_0^{2n+1} X \to \Omega_0^{2n+1} \operatorname{Im} J$ and assume that $n \geqq 1$ here. Then Lemma 6.2 implies that there is a primitive element in $H_{q-2d-2}(\Omega_0^{2n+1} X; F_p)$ of infinite height if $q = 2d - 2 > 0$. Thus Lemma 6.3 applies to give the theorem if $q - 2d - 2 > 0$. In case $q - 2d - 2 = 0$, then the first non-vanishing mod-p homotopy group of $\Omega_0^{2n+1} \operatorname{Im} J$ is in degree $2p - 2$ and Lemmas 6.2 and 6.3 apply to give the theorem.

Finally assume that $n = 1$. Thus there is a map $\gamma : P^{2p-3}(p) \to \Omega_0 X$ such that $(\Omega\Pi)(\gamma)$ represents a generator of $\Pi_{2p-3}(\Omega \operatorname{Im} J; F_p)$. Thus there are primitive elements v, u in degrees $2p - 3$ and $2p - 4$ such that u and v are mod-p spherical. Consider the polynomial algebra generated by $\beta Q_{p-1}^k(v)$ and notice that there is a sequence of Steenrod operations P_*^I with $P_*^I \beta Q_{p-1}^k(v) = e(\beta v)^{p^k}$, $e \neq 0$. This suffices and the theorem follows.

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